Simulation of the Catenary Effect Under Wind Disturbances
In Anchoring of Small Boats

by

Jessy Mbagara Mwarage

Submitted to the
Department of Mechanical Engineering
in Partial Fulfillment of the Requirements for the Degree of

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ABSTRACT

It has been conventional knowledge for as long as ships have existed that the catenary effect of an anchor line augments the efficiency of an anchoring system. This is achieved by making the anchor line as heavy as possible thus lowering the effective angle of pull on the anchor.

This notion has, however, come under criticism in recent times. Many small boat owners have shifted to lighter tauter lines for anchoring. The argument in favor of this new method is the cost savings associated with lighter anchoring and the tension relief that comes with using lighter and more elastic anchor lines.

The purpose of this study is to therefore compare the performance of long slack lines that form catenary shapes with that of shorter taut lines. An analysis is presented that describes the surge motion of a small anchored boat exposed to an input forcing function and various retarding forces and effects. The anchoring system used in the analytical model results in a non-linear but symmetrical restoring force, which resists the force-induced motion of the boat.

Two main types of anchor lines are considered: uniform-material and two-material anchor lines. Each anchor line is evaluated both in catenary configuration and taut configuration in terms of its ability to minimize the motions of the boat and tension force in the anchor line due to wind disturbances.

Supervisor: Douglas Hart
Title: Professor of Mechanical Engineering

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Chapter 1

Introduction: Project Motivation

An anchored vessel exposed to wind disturbances is a somewhat unique problem in dynamics. In most dynamic mechanical systems, the restoring force is a linear function of the displacement of the oscillating body and it is the response of this body to a time-varying disturbance force that is desired. The main objective is no different in the case of an anchored vessel; however, the restoring force of the anchor line is a nonlinear function of the displacement of the boat.

The purpose of this report is to consider the dynamic response of an anchored vessel exposed to wind disturbance. The class of vessels to be considered is small boats of the size usually used for pleasure where it is of interest to predict the surge motions of the boat (unidirectional motion along the boat's length) and the forces in the anchor line. The use of long slack lines that form catenary shapes is contrasted with the use of shorter taut lines. Both types of anchor lines are considered elastic, and non-linear, excursion-dependent restoring forces on the motions of the small boat. The major problems that must be solved by the engineer in designing such an anchoring system are to minimize the motion of the boat and tension in the anchor line due to wind disturbances. Therefore, picking the correct type of line (Catenary or Non-catenary) is important in ensuring maximum anchoring performance.

It has been conventional knowledge for as long as ships have existed that the catenary effect of an anchor line augments the efficiency of the anchoring system. This is achieved by lowering the effective angle of pull on the anchor by making the anchor line as heavy as possible. Therefore, the higher the degree of catenary, the greater the pull needed on the anchor line to straighten it out before it exerts any pull on the anchor.

This notion has come under criticism in recent times as evidenced in the following excerpt from a personal website on anchoring of small boats:

> This catenary has the convenient effect of lowering the effective angle of pull on the anchor ... the lore is [therefore] to use heavy chain behind the anchor ... Ships from all eras have used very heavy chain, and relatively small and ineffective anchors. This for the most part works well. Unfortunately, the relevant factors do not scale down evenly to smaller boats such as today's cruising and pleasure yachts. In fact, the best way to anchor a toy boat in the garden pond is with a relatively large toy anchor and a rode consisting entirely of an elastic line. Between this extreme and that of large ships, a compromise needs to be found. [3]

This study will therefore examine the veracity of the claims made in this excerpt and many like it by examining the responses of both catenary and non-catenary anchor line configurations to time-varying wind disturbances.

Chapter 2

Modeling the Anchoring Line Catenary

In this section the analytical equation used for catenary anchoring line is derived from first principles. The second order non-linear equation that describes the catenary is obtained and solved with the appropriate boundary conditions for a given anchor line configuration. Some simplifying assumptions that apply to a small boat situation and including a few applicable ones from Agarwal [4] were made for the analysis of catenary anchoring line as follows:

(a) The water body floor offers a rigid and frictionless support to the mooring line, which may lie on it,

(b) The anchor line moves very slowly inside the water so that any drag and inertial forces generated due to its motion are considered negligible,

(c) The water surrounding the anchor line is calm and therefore induces no change in the line geometry or in the line force due to direct fluid loading caused by waves and /or currents,

(f) The Anchor point is stationary through all time i.e. the magnitudes of disturbances are considered less than the force required to move the anchor,

(g) Only horizontal excursion of the catenary anchor line as a result of surge of the boat is considered.

2.1 The Uniform Material Catenary Equation

The following derivation of the catenary equation for a mooring line is based on the method used on Math24.net [1], with the exception of the boundary conditions applied.

Suppose that a chain of uniform mass per unit length is suspended at points *A* and *B*, which may be at different heights as in figure 1:

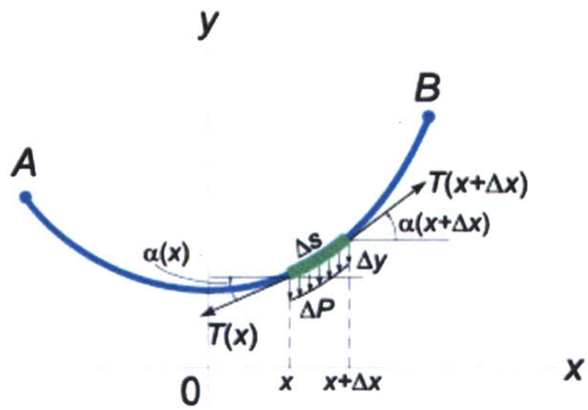


Figure 1: Infinitesimal Element in tension on a catenary line (adapted from [1])

In equilibrium, a small element of the chain of length Δs, has the distributed force of gravity acting on it given by:

$$\Delta P = \rho g A \Delta s \tag{1}$$

which in continuous form can be written as:

$$dP = \rho g A \cdot ds \tag{2}$$

where ρ is the density of the chain material, g is the acceleration of gravity, A is the cross sectional area of the chain. The tension forces thus generated are $T(x)$

and $T(x + \Delta x)$ which act at points x and $x + \Delta x$ respectively. The equilibrium conditions of the length element Δs in the x and y directions can be written as:

$$In\ the\ x - direction\colon\ -T(x) \cdot \cos(\alpha_x) + T(x + \Delta x) \cdot \cos(\alpha_{x+\Delta x}) = 0 \tag{3}$$

$$In\ the\ y - direction\colon\ -T(x) \cdot \sin(\alpha_x) + T(x + \Delta x) \cdot \sin(\alpha_{x+\Delta x}) - \Delta P = 0 \tag{4}$$

From the x-direction equilibrium equation, it follows that the horizontal component of the tension force, $T(x)$, is always a constant:

$$T(x) \cdot \cos(\alpha_x) = T(x + \Delta x) \cdot \cos(\alpha_{x+\Delta x}) = T_0 \tag{5}$$

From the y-direction equilibrium equation, a differential form of the equation can be obtained as follows:

$$\lim_{\Delta x \to 0} \left[\frac{T(x+\Delta x) \cdot \sin(\alpha_{x+\Delta x}) - T(x) \cdot \sin(\alpha_x)}{\Delta x}\right] = \lim_{\Delta x \to 0} \frac{\Delta P}{\Delta x} \tag{6}$$

$$\Rightarrow d(T(x) \cdot sin(\alpha_x)) = dP(x) \tag{7}$$

from which it follows that the tension of the cable as a function of the horizontal coordinate, x, can be written as:

$$T(x) = \frac{T_0}{\cos(\alpha_x)} \tag{8}$$

Plugging $T(x)$ from equation (8) into equation (7) yields:

$$d(T_0 \cdot tan(\alpha_x)) = dP(x) \ \Rightarrow\ T_0 \cdot d(tan(\alpha_x)) = dP(x) \tag{9}$$

Taking into account that the slope of the segment Δs is given by:

$$tan(\alpha_x) = \frac{dy}{dx} = y' \tag{10}$$

the equilibrium equation (9) can therefore be written as:

$$T_0 \cdot d(y') = dP(x) \tag{11}$$

Further, from equation (2), this equation can be written as:

$$T_0 \cdot d(y') = \rho g A \cdot ds \tag{12}$$

The arc length of an element of the chain, ds, can be expressed by the general formula for arc length as:

$$ds = \sqrt{1 + (y')^2} \cdot dx \quad (13)$$

From this equation and equation (12), the differential equation of the catenary can be written as:

$$T_o \frac{dy'}{dx} = \rho g A \sqrt{1 + (y')^2} \Rightarrow T_o y'' = \rho g A \sqrt{1 + (y')^2} \quad (14)$$

which is a non-linear second order differential equation. Therefore to solve it, the order of the equation can be reduced by using the first form of the equation (14), which can be solved by the separation of variables as:

$$\frac{dy'}{\sqrt{1+(y')^2}} = \frac{\rho g A}{T_o} dx \Rightarrow \int \frac{dy'}{\sqrt{1+(y')^2}} = \frac{\rho g A}{T_o} \int dx \quad (15)$$

Denoting $\frac{\rho g A}{T_o}$ as $\frac{1}{a}$ and computing the integral yields:

$$\ln\left(y' + \sqrt{1 + (y')^2}\right) = \frac{x}{a} + C_1 \quad (16)$$

where C_1 is a constant of integration. Therefore:

$$y' + \sqrt{1 + (y')^2} = C_1 \cdot e^{\frac{x}{a}} \quad (17)$$

To obtain $y(x)$, further integration is necessary, which is achieved by first multiplying both sides of the equation by the conjugate expression of the right-hand side: $y' - \sqrt{1 + (y')^2}$. This proceeds as follows:

$$\left(y' + \sqrt{1 + (y')^2}\right) \cdot \left(y' - \sqrt{1 + (y')^2}\right) = \left(y' - \sqrt{1 + (y')^2}\right) \cdot e^{\frac{x}{a}} \quad (18)$$

$$((y')^2 - (1 + (y')^2)) = \left(y' - \sqrt{1 + (y')^2}\right) \cdot e^{\frac{x}{a}} \quad (19)$$

$$-1 = \left(y' - \sqrt{1 + (y')^2}\right) \cdot e^{\frac{x}{a}} \quad (20)$$

$$y' - \sqrt{1 + (y')^2} = -C_1 \cdot e^{-\left(\frac{x}{a}\right)} \tag{21}$$

Adding equation (17) to equation (21) yields:

$$y' = C_1 \cdot \left[\frac{e^{\frac{x}{a}} - e^{-\left(\frac{x}{a}\right)}}{2}\right] = C_1 \cdot \sinh\left(\frac{x}{a}\right) \tag{22}$$

Integrating once more for $y(x)$ yields the general equation of the catenary as:

$$y(x) = C_1 \cdot a \cdot \cosh\left(\frac{x}{a}\right) + C_2 \tag{23}$$

where C_1 and C_2 are integration constants to be determined from the appropriate boundary conditions. Thus, for a mooring line with given boundary conditions: $y(0) = 0$ and $y(L) = H$, where L and H are the Length and Height to the point of attachment on the boat from the point of anchoring, the shape of the mooring line is described by:

$$y(x) = \left[\frac{H}{1 - \cosh\left(\frac{L}{a}\right)}\right] \cdot \left[1 - \cosh\left(\frac{x}{a}\right)\right] \tag{24}$$

where the shape of the catenary is uniquely determined by the shape parameter: $a = \frac{T_O}{\rho g A}$ which characterizes the material and geometrical properties of the chain and the external forces – Horizontal tension and Gravitational force – acting on it.

2.2 The Two-Material Catenary Equation

Given equation (23), plotting a catenary line that consists of two materials is now only a matter of choosing appropriate boundary conditions. Thus, for a two-material mooring line where the junction between the two materials is defined as L_j, which is the distance from the anchor at zero to the junction, two pairs of boundary conditions are required for each catenary section of the anchoring line. For the first section, these are:

$$y_1(0) = 0 \text{ and } y_1(L_j) = H_j, \tag{25}$$

where H_j is defined as the height of the junction from the water-body floor. To ensure continuity of the anchor line, the second section boundary conditions are defined as:

$$y_2(L_j) = H_j \text{ and } y_2(L) = H, \tag{26}$$

where L and H are the length and height to the boat as defined in section 2.1.

Applying these boundary conditions therefore yields the following piecewise equation for the two-material catenary:

$$y_1(x) = \left[\frac{H}{1-\cosh\left(\frac{L}{a_0}\right)}\right] \cdot \left[1 - \cosh\left(\frac{x}{a_0}\right)\right], for\ x \leq L_j \tag{27}$$

$$y_2(x) = C_1 \cdot a_1 \cdot \cosh\left(\frac{x}{a_1}\right) + C_2,\ for\ x \geq L_j \tag{28}$$

where C_1 and C_2 are defined as:

$$C_1 = \frac{H-H_j}{a_1 \cdot \left[\cosh\left(\frac{L}{a_1}\right) - \cosh\left(\frac{L_j}{a_1}\right)\right]} \tag{29}$$

$$C_2 = H_j \cdot \left\{1 + \frac{\cosh\left(\frac{L_j}{a_1}\right)}{\left[\cosh\left(\frac{L}{a_1}\right) - \cosh\left(\frac{L_j}{a_1}\right)\right]}\right\} - \frac{H \cdot \cosh\left(\frac{L_j}{a_1}\right)}{\left[\cosh\left(\frac{L}{a_1}\right) - \cosh\left(\frac{L_j}{a_1}\right)\right]} \tag{30}$$

and H_j is in turn defined as:

$$H_j = H \cdot \left[\frac{1-\cosh\left(\frac{L_j}{a_0}\right)}{1-\cosh\left(\frac{L}{a_0}\right)}\right] \tag{31}$$

Note that a_0 and a_1 capture the different material properties and geometric configuration of the two sections of the catenary. Recall from section 2.1 that it is defined as $\frac{T_o}{\rho g A}$ where T_o, the tautness of the undisturbed anchor line, describes geometry and $\rho g A$, the product of density, gravitational acceleration and effective cross-sectional area of the anchor line, describes material properties.

In this study, the two-material catenary is considered to be a steel chain connected to an anchor on one end and a nylon line on the other end; the other end of the nylon line is connected to the boat. The junction between the two lines is defined as being at half the horizontal distance from the anchor to the resting position of the boat. The effective axial stiffness of the anchor line is a weighted average (by length of each section when the boat is at rest) of the axial stiffness of each of the two lines.

Chapter 3

Modeling Boat Dynamics

In this section an analysis will be presented that describes the surge motion of a small anchored boat when exposed to an input forcing function and various retarding forces and effects. The anchoring system used in the analytical model results in a non-linear but symmetrical restoring force, which resists the force-induced motion of the boat. Only surge motion (boat displacement in the axis parallel to the bow-stern axis) is considered; motion in other degrees of freedom of the boat i.e. heave, sway, pitch, yaw, and roll, are considered negligible for the purpose of characterizing the catenary effect of the mooring line. Finally, the boat is treated as an ellipsoid body for the purpose of estimating the added mass of the boat. Therefore the differential equation describing the dynamic response of the boat can be expressed as:

$$\ddot{L} = \frac{1}{m_{TOT}}\{F_{IN}(t) - F_R(t) - F_D(t)\} \tag{32}$$

Each of the terms in the equation is explored in this section. It should be noted that the innovation in all this, which builds on those explored by Raichlen [5], concerns the incorporation of the full analytical model of a non-linear symmetrical restoring force that is dependent on the boat's excursion.

3.1 Input Force

As already stated, one of the forces acting on the boat is a time-dependent input force, $F_{IN}(t)$. This acts as the forcing function that governs the boat's dynamic response. In the current simulation, it is user-defined as one of two forcing functions considered to be possible wind profiles: the Heaviside function (step function) and the Ramp function. Each is designed so that the steady state position of the boat is the same in each case to ensure that dynamic responses are comparable in magnitude. As Agarwal [4] notes, it is customary in the design of off-shore mooring systems to use either a single design wave chosen to represent the expected extreme conditions in the area of interest or the statistical representation of waves during such conditions. Many of these waves involve oscillatory components. However, since the current study is focused on response to shock disturbances and not oscillatory ones, it is assumed that the step and ramp functions cover the general forms of shock disturbances that a small boat might experience at dock where the water is calm and that the responses obtained form a reasonable basis for the qualitative characterization of different anchoring systems.

3.2 Restoring Force

The restoring force acting on the boat is excursion-dependent i.e. its magnitude varies with the boat's displacement from its initial resting point. Raichlen [5] has examined the variation of the restoring force on a small boat with boat displacement as a function of the elastic characteristics of the mooring line. A similar concept is adopted albeit with one major difference: the full analytical (non-linear) solution for the restoring force is used as opposed to an approximation. It should be emphasized that the implicit assumption in this approach is that only surge of boat is being considered i.e. motion in other degrees of freedom of the boat (heave, sway, pitch, yaw, and roll) are considered negligible.

As in Raichlen [5], a region of free travel is incorporated in the forward excursion of the boat moored to a slack mooring line. This is a region of zero restoring force as the horizontal tension in the cable here is considered negligible in retarding the boat's motion; the horizontal tension is considered sufficient only to support the cable's catenary shape, an assumption that Raichlen proves to be empirically reasonable. This type of restraint is sketched in figure (2). Beyond a critical forward excursion point, L_{crit}, the restoring force is related to the geometry and the extension of the anchor line, which is assumed to be a linear elastic response for sufficiently small loads. By symmetry about the origin, a similar load-excursion relationship applies in the reverse direction of boat motion with the critical excursion point being $-L_{crit}$.

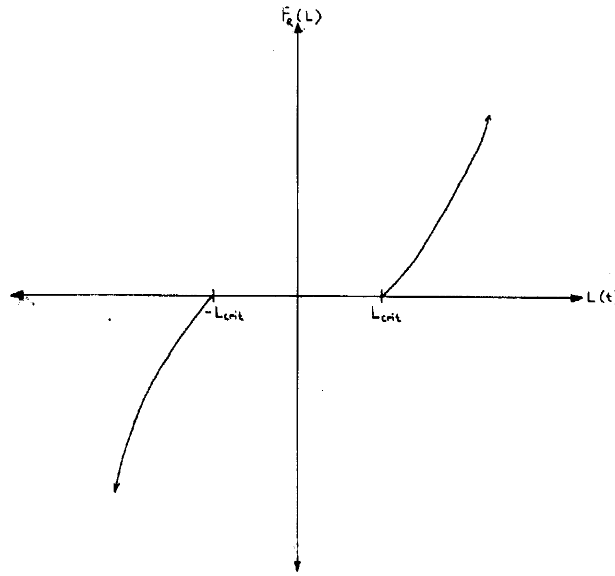


Figure 2: General form of restoration force of a catenary anchor line

Mathematically, the restoring force of the boat beyond the critical excursion point, L_{crit}, may be obtained from figure (3) as:

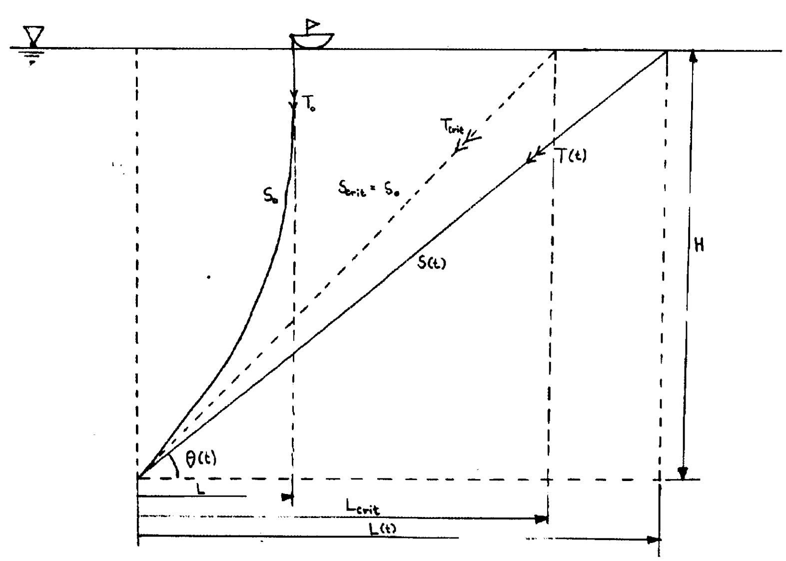


Figure 3: Geometry of anchor line at a given instance in time

where the restoring force, F_R, at a given instant in time, t, is given by:

$$F_R(t) = T_{line}(t) \cdot \cos(\theta(t)) \tag{33}$$

From linear elastic theory, the average tension along the line will be proportional to the strain of the line, ϵ, and the effective stiffness of the line, EA, where E is the Young's Modulus of the line material and A is the effective cross-sectional area of the line:

$$T_{line}(t) = \left[\frac{s(t)-s_0}{s_0}\right].EA \tag{34}$$

Notice that s_0 is just the original arc length of the mooring line as a rigid catenary is considered as discussed in section 2.2.1. On the other hand, $s(t)$ is the length of the mooring line at time t and is determined exactly from the geometry of the mooring line as:

$$s(t) = \sqrt{H^2 + L(t)^2} \tag{35}$$

Plugging equations (34) and (35) into equation (33) and considering the geometry of the boat at the time instant, t, the restoring force on the boat is fully expressed as:

$$F_R(L,t) = EA \cdot L(t) \cdot \left(\frac{1}{s_0} - \frac{1}{\sqrt{H^2+L(t)^2}}\right) \tag{36}$$

3.3 Hydrodynamic Damping Force

In the course of its motion, the boat naturally experiences a viscous drag force due to its interaction with water particles. Therefore, a viscous drag force, also considered a hydrodynamic damping force due can be defined in terms of a relative velocity as:

$$F_D(t) = \frac{1}{2}\rho_{water} C_D A \cdot (v_{boat} - v_{water})^2 \tag{37}$$

where ρ_{water} is the density of water, C_D is the drag coefficient for the boat in surge, A is the frontal area of the boat submerged in water and $v_{boat} - v_{water}$ is the net velocity of the boat. In differential form, $v_{boat} = \dot{L}$, which is the time derivative of the boat's excursion, captures $v_{boat} - v_{water}$ as the water is assumed to be calm. This means that the hydrodynamic drag force can finally be expressed as:

$$F_D(t) = \frac{1}{2}\rho_{water} C_D A \cdot \dot{L}^2 \tag{38}$$

which is yet another non-linear term to the boat's differential equation. The Coefficient of Drag, C_D, was obtained from the experimental results reported in [2] for 8 different types of ship hulls in the form of a graph shown below:

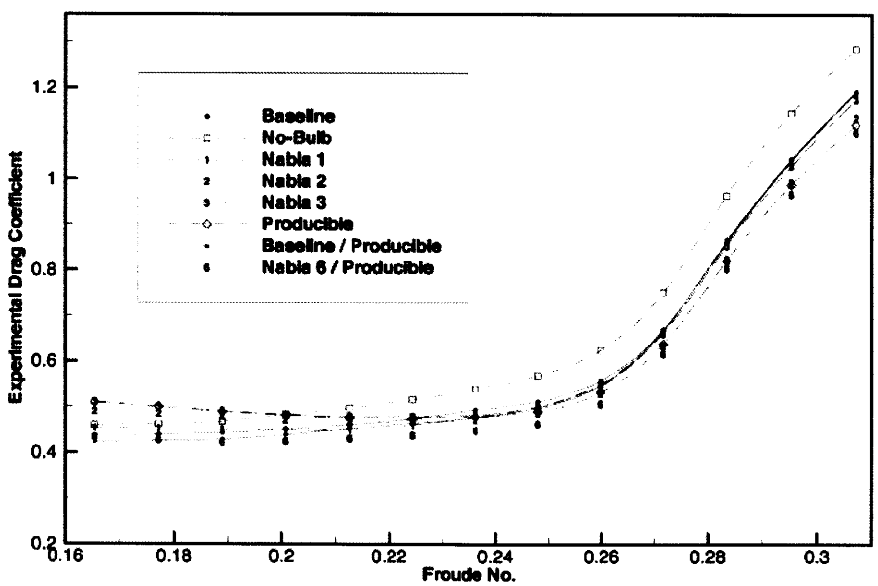


Figure 4: Variation of Drag Coefficient with Froude Number for 8 different types of ship hulls (adapted from [2])

Since the functional fits for each of the C_D were very close to each other, an approximate functional form of $C_D(Fr)$ - Coefficient of Drag as a function of the Froude Number - was derived from select data points in the graph and used in the boat simulation. The functional form derived was of the form:

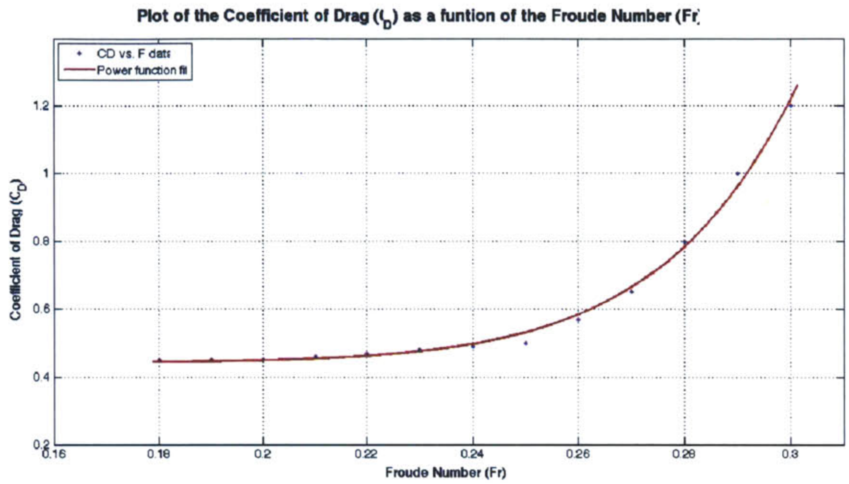


Figure 5: Fitted functional form to Drag Coefficient vs. Froude Number data

Where the general model fitted to the data was a power law of the form:

$$C_D(Fr) = A.Fr^B + C \tag{39}$$

and the coefficients A, B and C were obtain (with 95% confidence bounds) as:

$$A = 1.354\mathrm{e}+06 \ \ (-9.834\mathrm{e}+05, 3.691\mathrm{e}+06) \tag{40}$$

$$B = 11.93 \ \ (10.51, 13.36) \tag{41}$$

$$C = 0.442 \ \ (0.4221, 0.462) \tag{42}$$

with a goodness of fit given by an R-square value of 0.9947 and Root Mean Square Error of 0.01923. Therefore, the Coefficient of Drag is given as:

$$C_D(Fr) = 1.354\times10^6 . Fr^{11.93} + 0.442 \tag{43}$$

where it is assumed, reasonably, that $0 \leq Fr \leq 0.30$ which yields $0.442 \leq C_D \leq 1.2$ for the entire range of the boat's motion. Note that the Froude Number is defined as:

$$Fr \stackrel{\text{def}}{=} \frac{\dot{L}}{\sqrt{g.L_{boat}}} \tag{44}$$

where $\dot{L}$ is the velocity of the ship, g is the acceleration due to gravity and L_{boat} is the span of the boat at the waterline, assumed to be just the span of the boat.

On the other hand, the Frontal Area of the boat is approximated as:

$$A = \frac{V_{submerged}}{L_{boat}} \tag{45}$$

where $V_{submerged}$ is the submerged volume of the boat. Its value is obtained from Archimedes Principle from the known mass of the boat, m_{boat}, as:

$$V_{submerged} = \frac{m_{boat}}{\rho_{water}} \tag{46}$$

3.4 Added Mass Effect

The final force that acts on the boat in motion is an inertial force that is introduced by the acceleration and deceleration of the boat. This increasing and decreasing of the velocity profile affects the surrounding fluid. For example, if the fluid were at rest and the body were accelerated, then there would be a force opposing motion, other than the hydrodynamic damping force, caused by the body's accelerating of a portion of the surrounding fluid in the opposite direction. This additional force is conveniently represented as the product of the acceleration of the boat and an added hydrodynamic mass. In this study, it is assumed that the important relative acceleration, $\ddot{L} - a_{water}$, where a_{water} is the acceleration of the water due to the boat's motion, is captured in the net acceleration of the boat as simply $\ddot{L}$ given the assumption that the water is calm and therefore a_{water} is negligible. Therefore, the total force the boat experiences is expressed as:

$$m_{TOT}\ddot{L} = (m_{boat} + m_{added}).\ddot{L} \tag{47}$$

where m_{added} is computed for an ellipsoid (approximate shape of the boat's hull). According to Browning [6], this provides a reasonable first approximation of the important added mass term in surge. Therefore, consider the boat's hull as half of an ellipsoid where the full ellipsoid is described by the equation:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1 \tag{48}$$

where a, b and c are the semi-major, semi-minor and semi-vertical axes of the ellipsoid respectively corresponding to half the length, breadth and submerged height of the boat respectively.

Therefore, the added mass of the ellipsoid can be computed using the length, breadth and depth of the boat hull (*a*, *b* and *c* respectively) as:

$$m_{added} = \left(\frac{\alpha_0}{2-\alpha_0}\right) \cdot \frac{2}{3}\pi\rho_{water}abc \tag{49}$$

where $\frac{2}{3}\pi\rho_{water}abc$ is the mass of the volume of fluid displaced by the half of the ellipsoid that is considered submerged and α_0 parameter is a purely numerical quantity that describes the relative proportions of the ellipsoid. α_0 is defined as:

$$\alpha_0 = \left(\frac{2.(1-e^2)}{e^3}\right) \cdot \left[\frac{1}{2}\log\left(\frac{1+e}{1-e}\right) - e\right] \tag{50}$$

and the eccentricity, e, of the ellipsoid is defined as:

$$e = \sqrt{1 - \left(\frac{b}{a}\right)^2} \tag{51}$$

The depth of the boat submerged, c, is approximated from the breadth of the boat ($2b$) and the computed submerged frontal area of the boat (A) as:

$$c = \frac{A}{2b} \tag{52}$$

Finally, as noted by Browning [6], the added mass coefficients can vary greatly with changes in the depth of water, with increases reaching as much as twice the actual mass in shallow water. In the current study it is assumed that sufficiently deep water exists so that these effects can be ignored.

3.5 Differential Equation for Boat Motion

Therefore, considering all the effects accounted for in sections 3.1 to 3.4, the differential equation of motion of a small moored boat in surge is given by:

$$\ddot{L} = \frac{1}{m_{TOT}}\left\{F_{IN}(t) - EA \cdot L \cdot \left(\frac{1}{s_0} - \frac{1}{\sqrt{H^2+L^2}}\right) - \frac{1}{2}\rho_{water} C_D A \cdot \dot{L}^2\right\} \tag{53}$$

which can easily be solved numerically by one of MATLAB's ode-solvers. The solvers used in this study were the most accurate offered in the MATLAB environment to solve stiff differential equations i.e. ode45 and ode113.

Chapter 4

Simulation Results

A series of simulation runs were conducted on the dynamic system presented in chapter 3. This chapter examines the conditions of the various test cases and the qualitative and quantitative results that answer the central question: is catenary useful in rejecting shock disturbances in shallow water anchoring of small boats?

4.1 Qualitative Responses

For this study, two configurations of anchoring lines were considered:

a) Slack lines long enough to just form a catenary shape.

b) Taut lines long enough to just NOT form a catenary shape.

Also, two types of anchor likes were considered:

a) Uniform-material anchor lines.

b) Two-material anchor lines.

All possible permutations of the above four types of lines were considered. Each type of line was subjected to both a step and a ramp input force. The step input force was an instantaneous rise from 0 kg-f to 1500 kg-f while the ramp input force was a linear rise from 0 kg-f to 1500 kg-f in ten seconds. The parameters of the boat used in each case to define the physical properties of the boat and its interaction with the water body are given in the following MATLAB code excerpt:

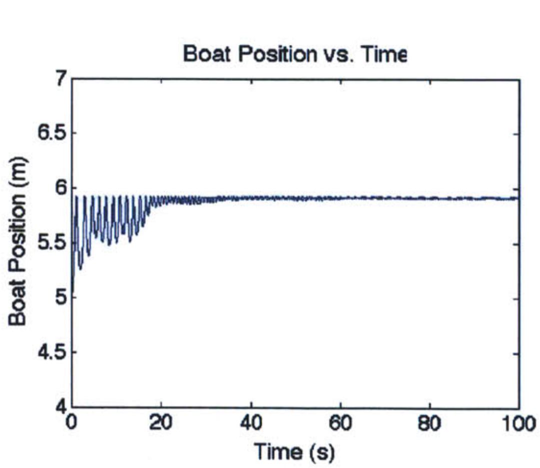
```
m_boat=6672; % Mass of boat in 'kg' - NOTE: Mass of boat used
here is 14700 lbs-f
g=9.8; % Gravitational acceleration in 'm/s/s'
l_boat=10.97; % Span of boat in 'm' - NOTE: Span of boat used
here is 36 ft
w_boat=3.66; % Breadth of boat in 'm' - NOTE: Breadth of boat
used here is 12 ft
rho_water=1025; % Average density of sea water in 'kg/m^3'
v_sub=m_boat/rho_water; % Submerged volume of boat from
Archimedes Principle in 'm^3'
A_frontal=v_sub/l_boat; % Frontal area of boat (for Drag
calculations) in 'm^2'
h_sub=A_frontal/w_boat; % Submerged height of boat (for Added
mass calculations) in 'm'
ecc=sqrt(1-(w_boat/l_boat)^2); % Eccentricity of an ellipsoid
used in calculating mass, m_added
alpha_0=((2*(1-ecc^2))/(ecc^3))*((0.5*log((1+ecc)/(1-ecc)))-ecc);
% Numerical constant used in calculating mass, m_added
m_added=0.5*((alpha_0)/(2-
alpha_0))*((4/3)*pi*rho_water*l_boat*w_boat*h_sub); % Added mass
of boat (resulting from inertial effects of water) in 'kg
m_tot=m_boat+m_added;% Effective mass of boat (including inertial
effects of water) in 'kg
Fr=0.289; % Maximum Froude Number used in calculating Coefficient
of drag i.e. at v_max=3m/s, dimensionless
CD=0.442+((1.354*10^6)*(Fr)^11.93); % Coefficient of drag used in
drag calculations, dimensionless
```

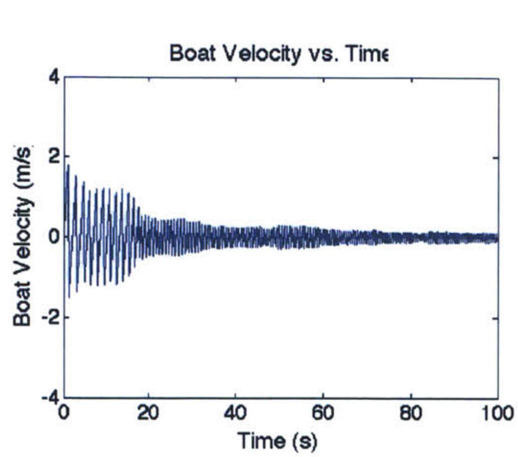
4.1.1 Uniform-material Anchor Lines

The Catenary anchor line chosen for the typical case had the following properties:

```
H=10; % Vertical height to boat in 'm'
L=5; % Horizontal length to boat in 'm'
rho_chain=7850; % Density of anchor chain in  'kg/m^3'
E_chain=200*10^9; % Young's modulus of chain in 'Pa'
D_chain=0.025; % Effective diameter of chain 'in m'
```

The qualitative responses to a step input in force were:





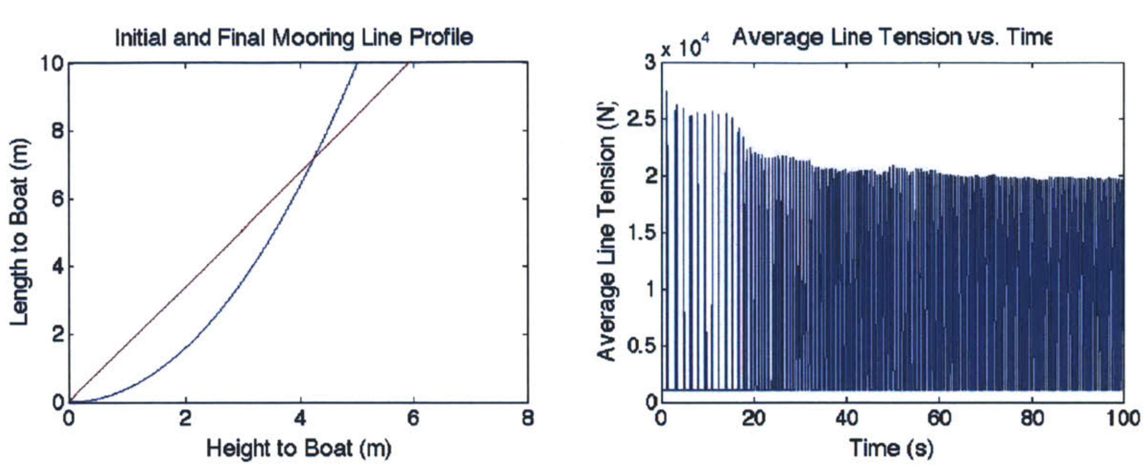


Figure 6: Typical dynamic response to a STEP input force of a boat anchored by a uniform material catenary-forming anchor line

The qualitative responses to a ramp input in force were:

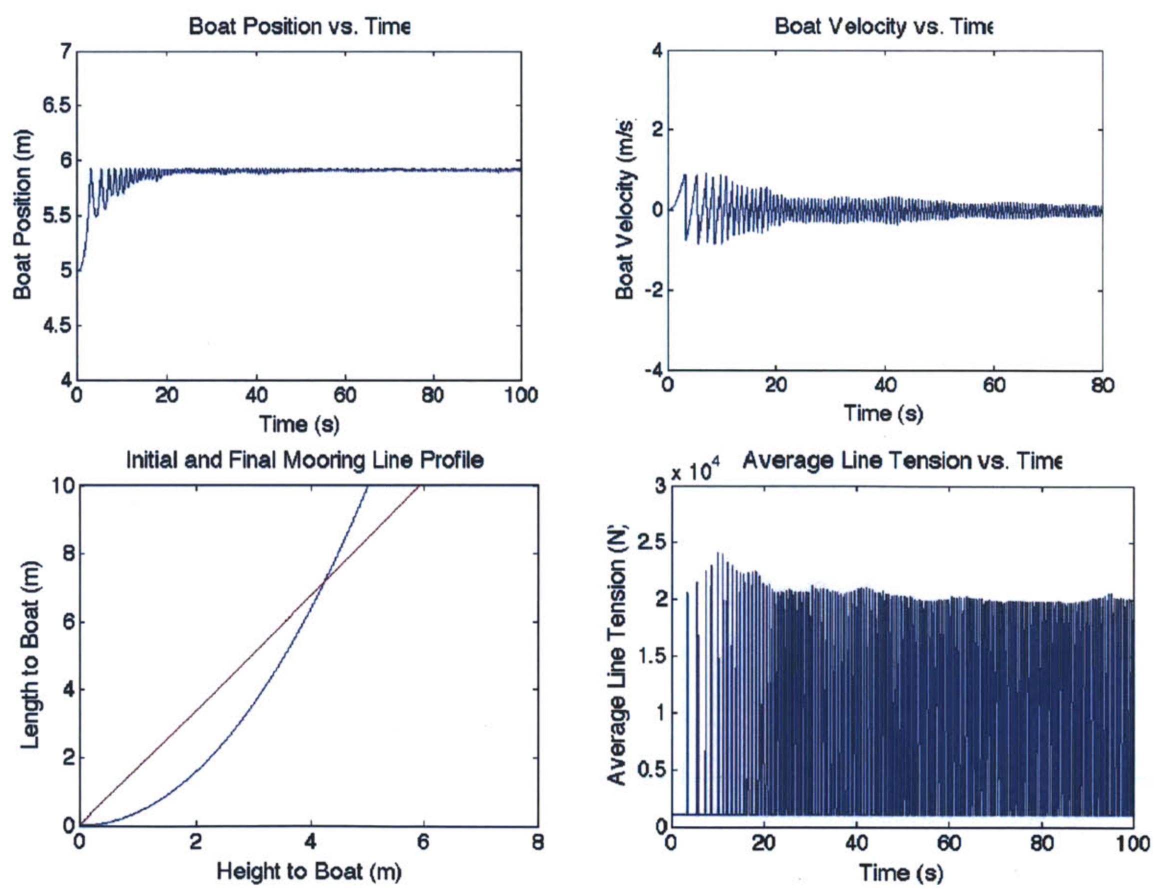


Figure 7: Typical dynamic response to a RAMP input force of a boat anchored by a uniform material catenary-forming anchor line

On the other hand, the Non-Catenary Mooring line chosen for the typical case had similar configuration but was taut enough to just form a straight line between boat and anchor:

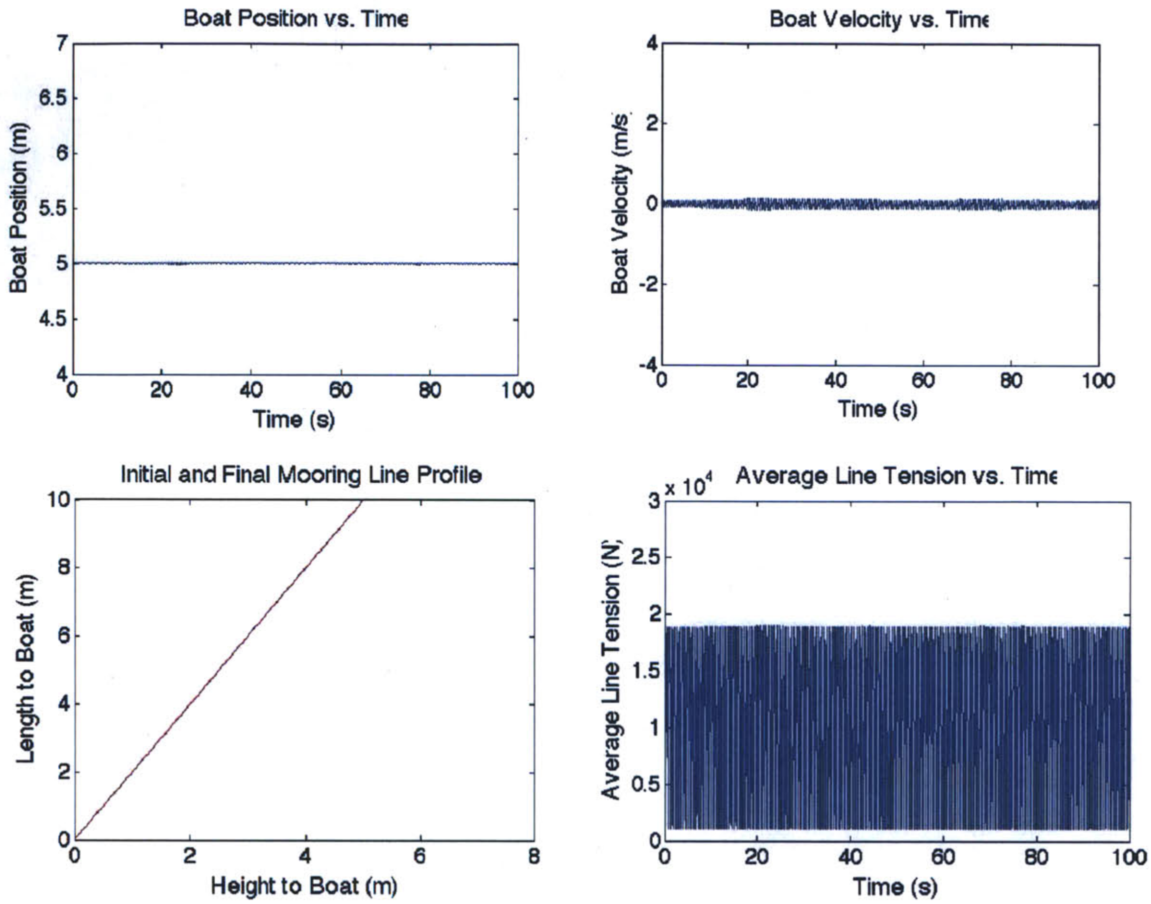
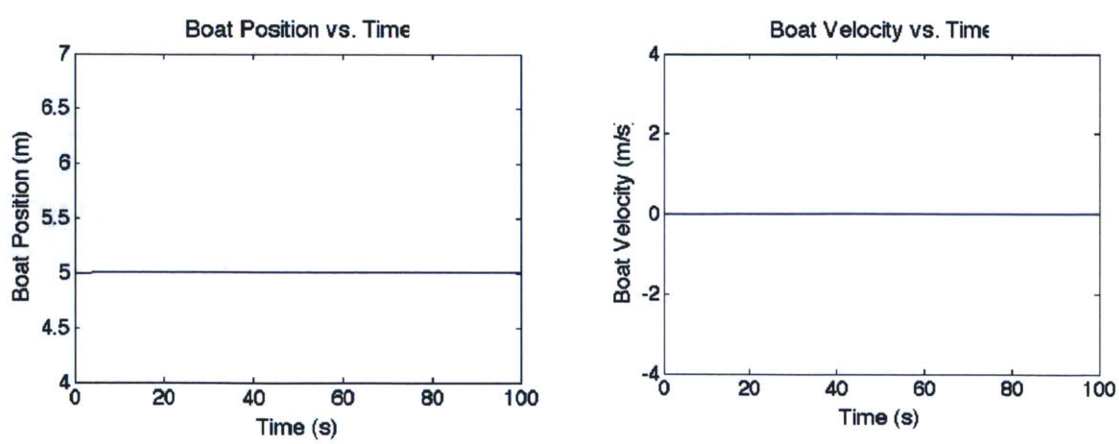


Figure 8: Typical dynamic response to a STEP input force of a boat anchored by a uniform material NON-catenary-forming anchor line

The qualitative responses to a ramp input in force were:



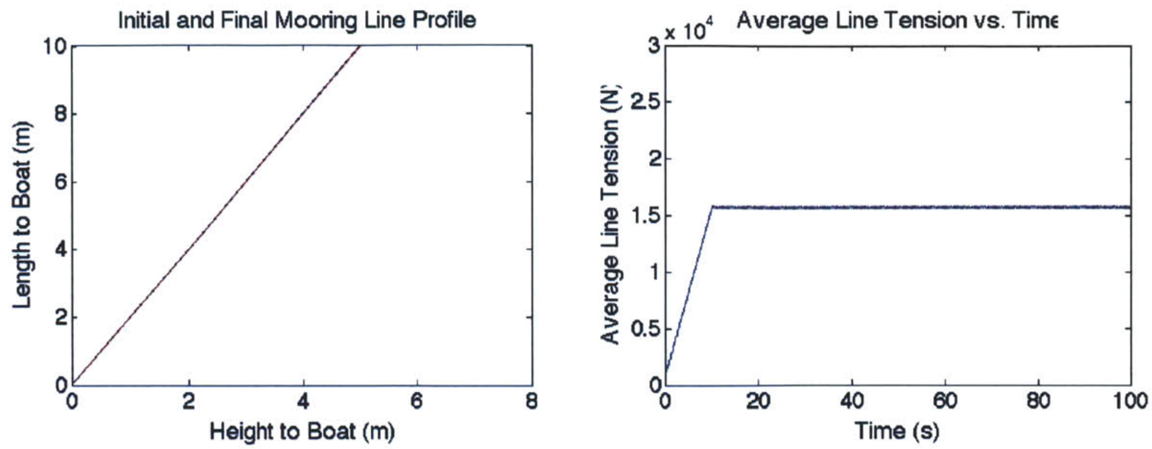


Figure 9: Typical dynamic response to a RAMP input force of a boat anchored by a uniform material NON-catenary-forming anchor line

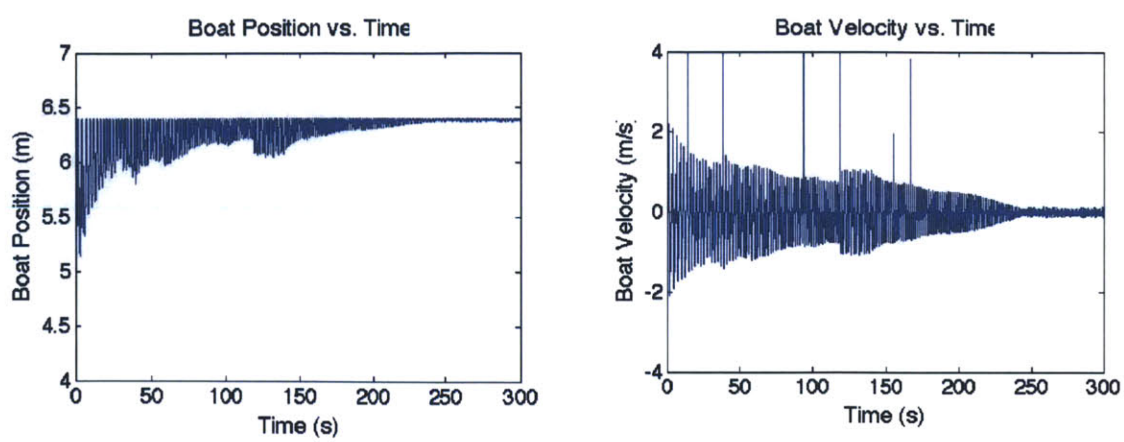
4.1.2 Two-material Anchor Lines

The Catenary anchor line chosen for the typical case here had the following properties

(MATLAB code excerpt defining both configuration and material properties):

```
H=10; % Vertical height to boat in 'm'
L=5; % Horizontal length to boat in 'm'
rho_chain_S=7850; % Density of STEEL anchor chain in 'kg/m^3'
rho_chain_N=1150; % Density of NYLON anchor chain in 'kg/m^3'
E_chain_S=200*10^9; % Young's modulus of STEEL chain in 'Pa'
E_chain_N=4*10^9; % Young's modulus of NYLON chain in 'Pa'
D_chain_S=0.03; % Effective diameter of STEEL chain 'in m'
D_chain_N=0.025; % Effective diameter of NYLON chain 'in m'
L_j=L/2; % Length to junction of STEEL and NYLON chains in 'm'
```

The qualitative responses to a step input in force were:



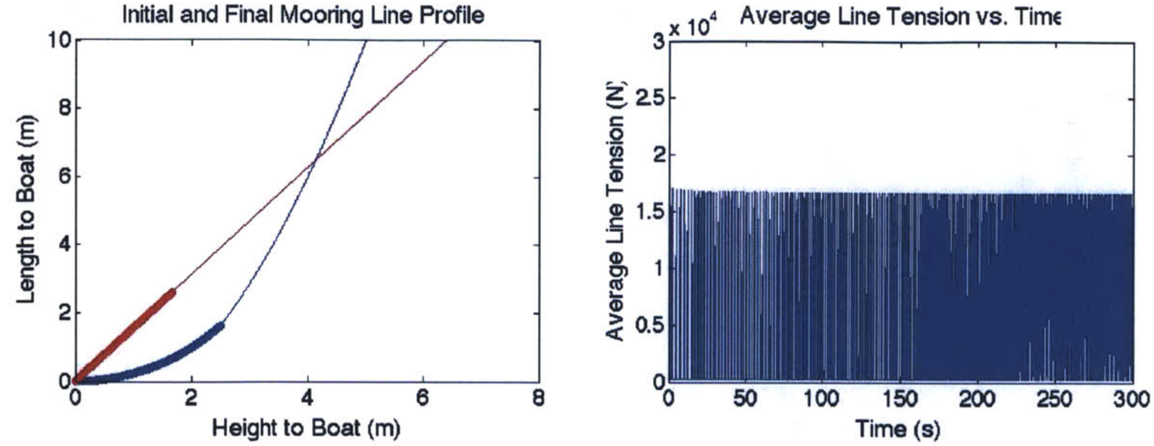


Figure 10: Typical dynamic response to a STEP input force of a boat anchored by a two-material catenary-forming anchor line

The qualitative responses to a ramp input in force were:

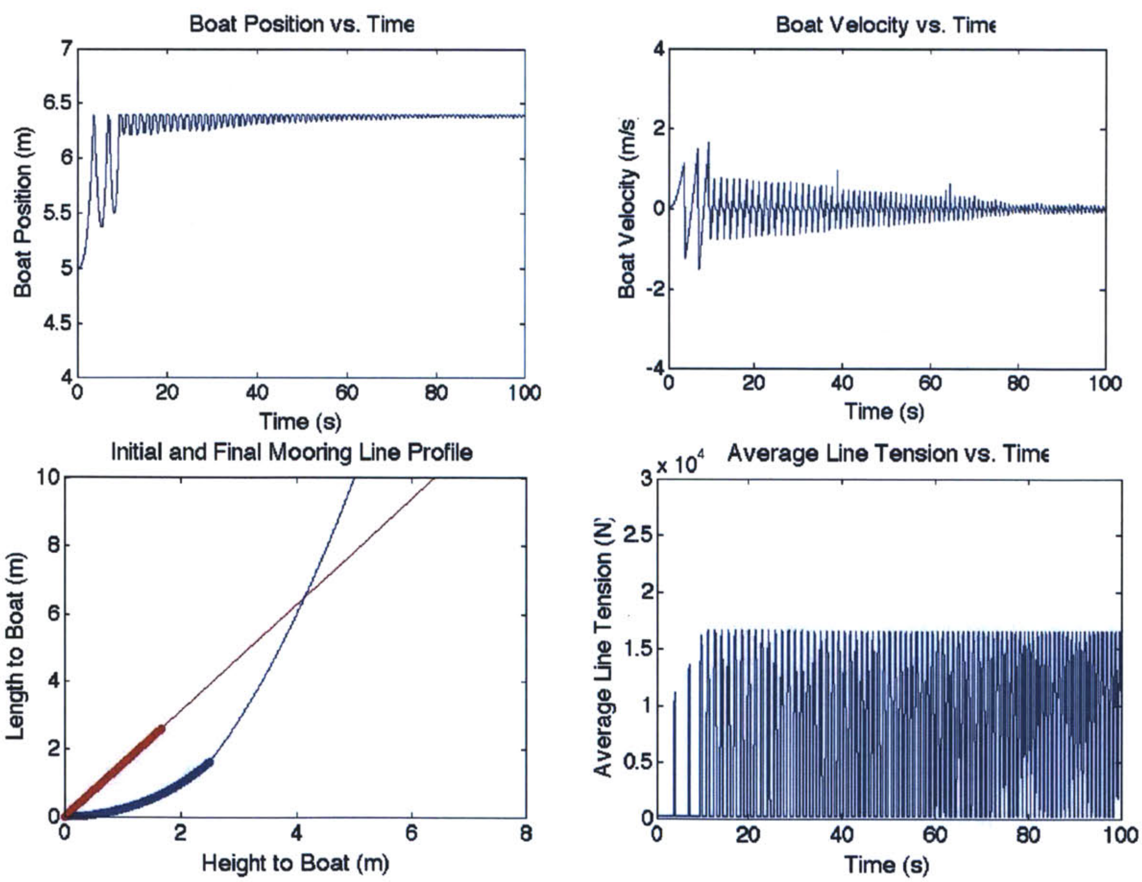


Figure 11: Typical dynamic response to RAMP input force of a boat anchored by a two-material catenary-forming anchor line

The Non-Catenary anchor line exhibited the following responses to a step input in force:

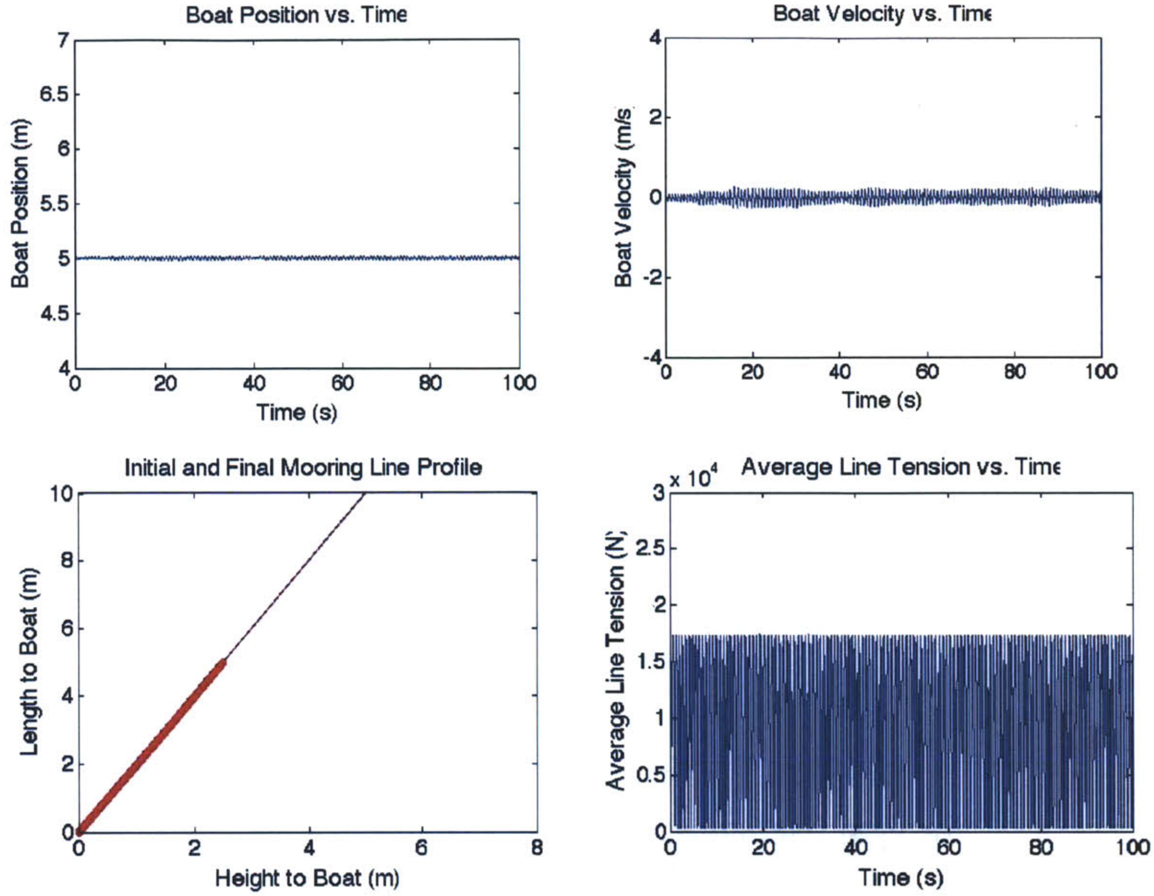
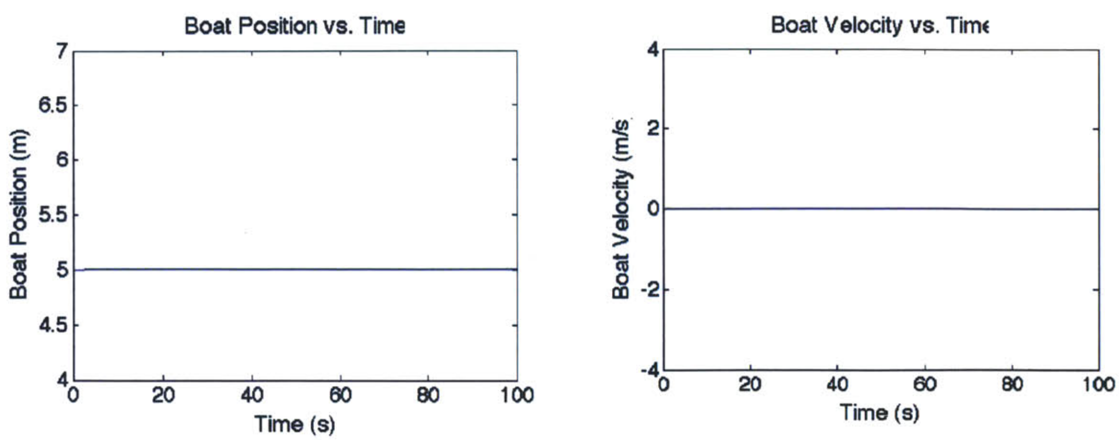


Figure 12: Typical dynamic response to a STEP input force of a boat anchored by a two-material NON-catenary-forming anchor line

The qualitative responses to a ramp input in force were:



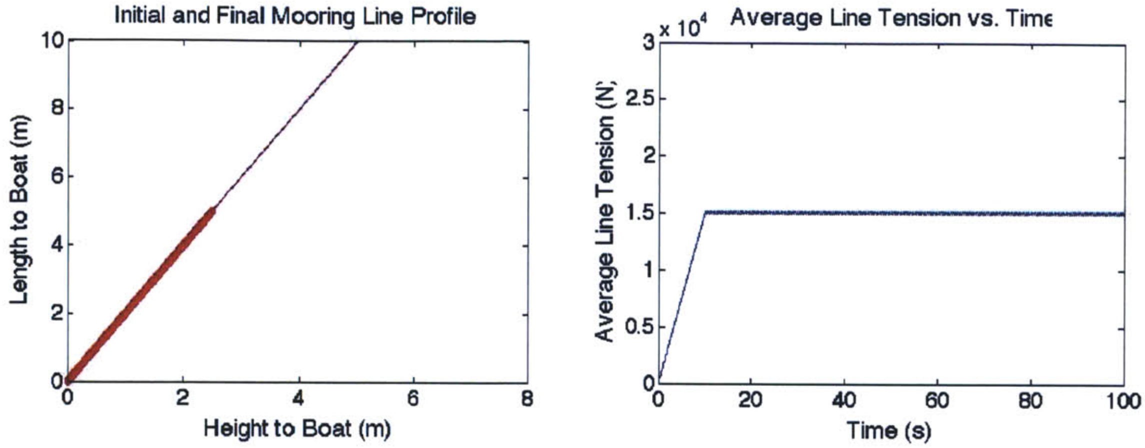


Figure 13: Typical dynamic response to a RAMP input force of a boat anchored by a two-material NON-catenary-forming anchor line

It is clear from the responses recorded above that oscillations of the boat increase significantly with the slackness of the anchor line. Therefore, catenary lines tend to increase the settling time of a boat's response to a wind disturbance. Also, it is clear that maximum tension achieved in a disturbance is higher in catenary lines due to the overshoot phenomenon experienced early in the boat's motion. It should not be forgotten however that a little slack in an anchor line is advantageous in rejecting small disturbances (smaller than those used in this study) due to the decreased angle of pull. This presents a trade-off for the design engineer: to balance the advantage of tension relief gotten from a tauter line with the advantage of lower angle of pull gotten from a slacker line.

4.2 Effect of Anchoring Line Scope

In light of the results from section 4.1, the most important factor to probe for minimizing maximum tension in the anchor line is scope: the ratio of the length of anchor line to water depth. This traces the variation in maximum tension with the slackness of the anchor line i.e. the higher the scope, the more slack the anchor line. This was investigated for a step input force to a boat (representative of the trend in a ramp input as well) anchored with a uniform-material anchor line and a two-material anchor line.

4.2.1 Uniform-material Anchor Lines

The maximum tension was thus found to vary linearly with changing scope:

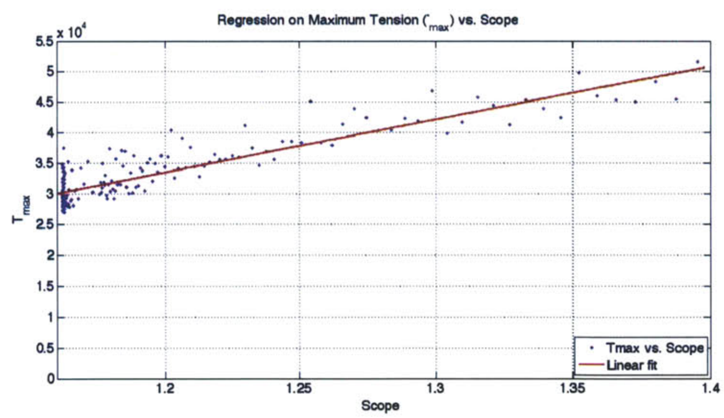


Figure 14: Regression on Maximum Tension vs. Scope for a uniform material catenary anchor line

Where the general model fitted to the data was linear of the form:

$$T_{max}(s) = Ms + C \tag{54}$$

and the coefficients M and C were obtain (with 95% confidence bounds) as:

$$M = 8.648\text{e} + 04 \;\; (8.029\text{e} + 04, 9.267\text{e} + 04) \tag{55}$$

$$C = -7.031\text{e} + 04 \;\; (-7.773\text{e} + 04, -6.288\text{e} + 04) \tag{56}$$

with a goodness of fit given by an R-square value of 0.8056 and Root Mean Square Error of 2331.

Clearly, the larger the scope (the longer the catenary for a given water depth and length to boat), the larger the maximum tension reached due to overshoot of the boat. Therefore, for uniform material anchor lines, there is an advantage to minimizing scope (making the anchor line as taut as possible) in order to minimize the maximum tension reached in the case of a step wind disturbance. A taut uniform-material anchor line performs better than a slack two-material anchor line in response to large wind disturbances.

4.2.2 Two-material Anchor Lines

The maximum tension here was found to be fairly constant with changing scope:

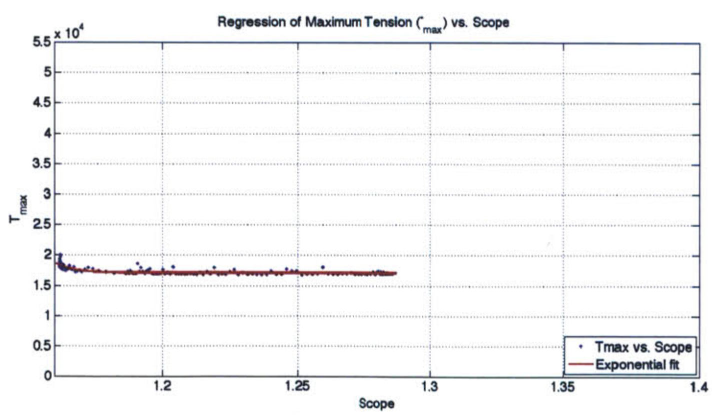


Figure 15: Regression on Maximum Tension vs. Scope for a two-material catenary anchor line

Where the general model fitted to the data was linear of the form:

$$T_{max}(s) = Ms^{B} + C \tag{57}$$

and the coefficients M, B and C were obtained (with 95% confidence bounds) as:

$$M = 1.3\text{e} + 16 \ \ (-1.577\text{e} + 17, 1.837\text{e} + 17) \tag{58}$$

$$B = -199.9 \ \ (-287.3, -112.6) \tag{59}$$

$$C = 1.711\text{e} + 04 \ \ (1.704\text{e} + 04, 1.719\text{e} + 04) \tag{60}$$

with a goodness of fit given by an R-square value of 0.6832 and Root Mean Square Error of 358.9.

Therefore, for two-material anchor lines, there is no advantage to minimizing scope (making the anchor line as taut as possible) in order to minimize the maximum tension reached in the case of a step wind disturbance. A taut two-material anchor line performs as well as a slack two-material anchor line. However as seen in section 4.2, minimizing scope reduces boat oscillations significantly.

Chapter 5

Conclusions and Recommendations

The following major conclusions and recommendations on the anchoring of small boats in shallow water may be drawn from this study:

1. For forced motions of small boats the characteristics of the restoring force are extremely important. Relatively small slack in an anchor line can significantly increase the oscillations of a small boat but also has the effect of reducing angle of pull on the anchor, an important tradeoff in the use of uniform-material lines.

2. The elastic characteristics of an anchor line are equally important in determining the oscillations of a small boat. It is therefore recommended that if a two-material anchor line is used to anchor a small boat, that it be taut to reduce the oscillation periods of the boat with the advantage of maximum tension relief. Where a uniform-material anchor line slack line has to be used, it is recommended that it be a little slack to reduce angle of pull on the anchor.

3. The approach used in predicting the dynamic response of small moored boats has been described in some detail in this study. It is recommended that physical experiments be carried out on a small boat to validate the models employed.

Chapter 6

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